My research interests are categorification, low dimensional topology, and representation theory. I am particularly interested in homology theories in low dimensional topology, and their categorical / representation theoretic foundations. I am driven by the rich structures afforded by link homology theories such as Khovanov homology and HOMFLY-PT homology, and their deep connections with geometric representation theory and algebraic geometry.

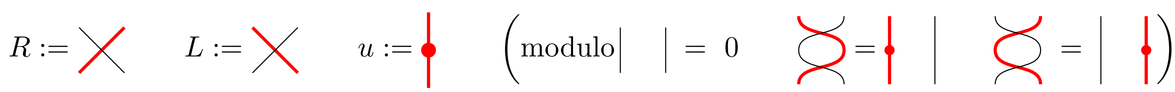
0.1. **Graphical calculus and knot Floer homology.** In [Web13], Webster defined a family link homology theories, associated to an arbitary complex semisimple Lie algebra $\mathfrak{g}$. Webster's homology is isomorphic to the previously constructed Khovanov homology, respectively Khovanov–Rozansky homology, when $\mathfrak{g} = \mathfrak{gl}_2$ (respectively $\mathfrak{g} = \mathfrak{gl}_N$).

It is conjectured [DGR06] that there should also be link homology theories associated to certain Lie superalgebras, for instance Knot Floer homology should be $\mathfrak{gl}_{1,1}$ link homology. Unfortunately, it Webster's theory does not apply to Lie superalgebras, and it remains an open problem to construct $\mathfrak{gl}_{N,M}$ link homology theories for $M, N \geq 1$ (the case $M = 0$ or $N = 0$ being accomplished already). On the other hand, Ozsvath–Szabo have obtained a purely algebraic description of knot Floer homology, in terms of their so-called "bordered knot algebras" [OS16; OS17].

During a reading course on bordered knot algebras, with my graduate student Connor Gallagher, it was realized that we could give a Webster-like flavor to these bordered knot algebras, and doing so revealed an action of Soergel bimodules that was not noticed before.

The main player is the following monoidal category that we introduce.

Definition 0.1. Let $\mathcal{C}$ be the $\mathbb{Z}$-graded monoidal category generated by objects $\mathfrak{r}$ (*red*) and $\mathfrak{b}$ (*black*), with generating morphisms (and relations)



Here we are employing the usual diagrammatics for discussing morphisms in monoidal categories, reading from bottom to top, so that for instance R maps $\mathfrak{rb} \to \mathfrak{br}$.

Remark 0.2. The above graphical calulus looks like a hybrid of the "redotted Webster algebras" from [Kho+22], and the homology of tangle Floer algebras from [PV14].

Theorem 0.1 (In preparation, joint with grad student Connor Gallagher)**.** *Let $\mathcal{C}(m,k)$ be the full subcategory of $\mathcal{C}$ consisting of objects with exactly m occurences of $\mathfrak{r}$ and k occurences of $\mathfrak{b}$. Let $A_{SOS}(m,k)$ denote the endomorphism algebra of the additive generator of $\mathcal{C}(m,k)$. Then there is an isomorphism $\mathcal{B}(m,k,\emptyset) \cong A_{SOS}(m, m+1-k)$, where $\mathcal{B}(m,k,\emptyset)$ is the algebra defined by Oszvath-Szabo in [OS16].*

Theorem 0.2 (In preparation, joint with grad student Connor Gallagher)**.** *There is an action of the category SBim_n of Soergel bimodules on $\mathcal{C}(n,k)$ such that the induced categorical braid group action given by Rouquier complexes is quasi-isomorphic to the action constructed in [OS16].*

Remark 0.3. Theorem 0.2 is useful because it realizes the action of the braid group on $\mathcal{C}(m,k)$ by dg functors, rather than A_∞-functors.

There are a lot of future directions to pursue here, and at least one of them of them will surely be the subject of Connor's Ph.D. research. Of these, the following is quite important.

Problem 0.4. Further elucidate the relationship between Soergel bimodules and knot Floer homology, obtaining a spectral sequence from triply-graded Khovanov–Rozansky homology to knot Floer homology. This would prove a conjecture of Dunfield–Gukov–Rasmussen [DGR06].

0.2. **Soergel bimodules, Hilbert schemes, and representabily of link homology.** Another project of mine with concerns the construction of objects which represent link homology theories, much like how Eilenberg–Maclane spaces represent cohomology groups of spaces, in classical algebraic topology. These representing objects ought to reveal some higher structure, in the same way that the Steenrod algebra of stable cohomology operations appears as the endomorphism algebra of the relevant Eilberg–Maclane spectrum.

For the purposes of this research statement I am concerned only with link homology theories in "type A", and the representing objects we seek will be certain complexes of Soergel bimodules. Instead of Steenrod algebras, in this context we expect to find Rasmussen differentials and deep connections with the Hilbert scheme of points in $\mathbb{C}^2$.

Since link homology functors are "trace-like" we start with an investigation of the object which represents the *universal* trace functor.

In work in progress I show how to construct a complex $\mathcal{PS} \in \mathrm{Ch}^b(\mathrm{SBim}_n)$ that I call the *Procesi–Springer* complex because it places a role analogous to the Springer sheaf in character sheaves, and the Procesi bundle on the Hilbert scheme. This complex satisfies a number of remarkable properties.

Theorem 0.3 (In preparation)**.** *We have* $\mathrm{Hom}(\mathcal{PS}, \mathbb{1}) \simeq \mathrm{HH}_\bullet(\mathrm{SBim}_n)$ *and* $\mathrm{Hom}(\mathbb{1}, \mathcal{PS}) \simeq \mathrm{HH}^\bullet(\mathrm{SBim}_n)$*.*

It is known that $\mathrm{HH}_\bullet(\mathrm{SBim}_n)$ is formal [HL23] and isomorphic to the wreath product algebra $\Bbbk[\mathbf{x}, \boldsymbol{\vartheta}]\#S_n$ where $\mathbf{x} = \{x_1, \ldots, x_n\}$ are even (polynomial) variables and $\boldsymbol{\vartheta} = \{\vartheta_1, \ldots, \vartheta_n\}$ are odd (exterior) variables [GHW21]. On the other hand, computing Hochschild cohomology $\mathrm{HH}^\bullet(\mathrm{SBim}_n)$ is still an open problem (essentially equivalent to the problem of computing $\mathrm{HH}^\bullet$ of Soergel's graded version of BGG category $O(\mathfrak{gl}_n)$).

In [Gor+], my coathors and I proved that $\mathrm{Ch}^b(\mathrm{SBim}_n)$ admits a Serre functor, given by tensoring with FT^{-1}, where FT is the Rouquier complex of the full twist braid. In terms of $\mathcal{PS}$ this has the following manifestation.

Theorem 0.4 (In preparation)**.** *The dual of* $\mathcal{PS}$ *satisfies* $\mathcal{PS}^\vee \simeq \mathrm{FT}^{-1} \star \mathcal{PS}$*, in particular* $\mathrm{Hom}(\mathcal{PS}, \mathrm{FT}) \simeq \mathrm{HH}^\bullet(\mathrm{SBim}_n)$*.*

This should be compared to the familiar fact that the Procesi bundle $\mathcal{P}$ on $\mathrm{Hilb}_n(\mathbb{C}^2)$ satisfies $\mathcal{P}^\vee \cong O(-1) \otimes \mathcal{P}$. The following says that $\mathcal{PS}$ represents the universal trace functor.

Theorem 0.5 (In preparation)**.** *The complex* $\mathcal{PS}$ *has the structure of a coalgebra object in* $\mathrm{Ch}^b(\mathrm{SBim}_n)$*, as well has an object in the Drinfeld center of* $\mathrm{Ch}^b(\mathrm{SBim}_n)$*, so that* $\mathrm{Hom}(\mathcal{PS}, -)$ *induces a functor from the horizontal trace* $\mathrm{hTr}(\mathrm{SBim}_m)$ *to the derived category of modules over* $\mathrm{Hom}(\mathcal{PS}, \mathbb{1}) \simeq \Bbbk[\mathbf{x}, \boldsymbol{\vartheta}]\#S_n$*. This functor is quasi-fully faithful, hence is quasi-equivalence onto its image. This image is the perfect derived category of* $\Bbbk[\mathbf{x}, \boldsymbol{\vartheta}]\#S_n$*.*

Remark 0.5. The relation between $\mathrm{hTr}(\mathrm{SBim}_n)$ and modules over the wreath product algebra was known already [HL23; GHW21], but the statements about $\mathcal{PS}$ are new.

There are a great many applications of $\mathcal{PS}$ that I plan to investigate in future work.

Problem 0.6. Using the S_n action on $\mathcal{PS}$, for each $0 \leq k \leq n$ define a direct summand $E_k \subset \mathcal{PS}$ to be the sgn $\times$triv isotypic component with respect to the restricted $S_k \times S_{n-k}$ action. Show that $\mathrm{Hom}(E_k, -)$ is isomorphic to the a-degree k part of triply graded Khovanov–Rozansky homology.

Remark 0.7. In a parallel project, joint with Ben Elias, we are pursuing much more explicit formulae for E_k. So far, we have succeeded in giving an explicit description E_1 using some new dg diagrammatics for Rouquier complexes.

Conjecture 0.8. The object $E_\bullet = \bigoplus_{k=0}^n E_k$ has an action of an exterior algebra in variables $d_{N,M}$ for $N, M \geq 0$, so that adding $d_{N,M}$ to the differential of $E_\bullet$ results in the object which represents $\mathfrak{gl}_{N,M}$ link homology.

0.3. **Invariants of manifolds coming from link homology.** Perhaps the biggest open problem in the subject of link homology / categorification is to construct categorified TQFT-like invariants of manifolds in dimensions ≤ 4.

There is a construction, essentially due to Bar-Natan which associates a vector space $\mathrm{BN}(Y, L)$ to each pair (Y, L) where Y is a compact oriented 3-manifold and $L \subset \partial Y$ is a closed 1-submanifold of the boundary. In slightly more detail, $\mathrm{BN}(Y, L)$ is formally spanned by embedded surfaces in Y with boundary L, modulo some local relations.

Now, if Σ is an oriented surface and $\mathbf{p} \subset \partial\Sigma$ is a finite set of points, then we have the *Bar-Natan category* $\mathcal{BN}(\Sigma, \mathbf{p})$; objects of this category are given by embedded 1-submanifolds $T \subset \Sigma$ with $\int T \subset \int \Sigma$ and $\partial T = \mathbf{p}$, and $\mathrm{Hom}_{\mathcal{BN}(\Sigma,\mathbf{p})}(S, T)$ defined to be the BN-module associated to the thickened surface $\Sigma \times I$ with boundary condition $T \times \{1\} \cup B \times I \cup S \times \{0\}$.

Fix a commutative ring $\mathbb{k}$. Let Y be an oriented 3-manifold and $L \subset \partial Y$ a closed 1-submanifold of the boundary. Associated to the pair (Y, L) we have the *Bar-Natan module* $\mathrm{BN}(Y, L)$, which is defined to be the $\mathbb{k}$-module spanned by isotopy classes of surfaces embedded in Y with boundary L, modulo some local relations.

Now, if Σ is an oriented surface and $\mathbf{p} \subset \partial\Sigma$ is a finite set of points, then we have the *Bar-Natan category* $\mathcal{BN}(\Sigma, \mathbf{p})$; objects of this category are given by embedded 1-submanifolds $T \subset \Sigma$ with $\mathrm{int}\, T \subset \mathrm{int}\, \Sigma$ and $\partial T = \mathbf{p}$, and $\mathrm{Hom}_{\mathcal{BN}(\Sigma,\mathbf{p})}(S, T)$ defined to be the BN-module associated to the thickened surface $\Sigma \times I$ with boundary condition $T \times \{1\} \cup B \times I \cup S \times \{0\}$.

In [**<empty citation>**] Rose, Wedrich and I show how to construct a dg category $\mathcal{FS}(\Sigma, \mathbf{p})$ which should be thought of as an appropriately derived version of the Bar-Natan category $\mathcal{BN}(\Sigma, \mathbf{p})$. In work in progress we are working on the the following problem.

Problem 0.9. Construct a dg (or A_∞) functor $\mathcal{F}(\Sigma', \mathbf{p}') \to \mathcal{F}(\Sigma, \mathbf{p})$ associated to an oriented cobordism (with corners) $Y \colon \Sigma' \to \Sigma$.

To achieve this, it would suffice to construct functors associated to 3-dimensional 0,1,2,3-handles and check the necessary relations. We have well-defined functors for handles of index 0,1,2, and an idea for how 3-handles should work (with details still being developed).

By comparing the construction with the Justin Roberts skein theoretic description of Turaev–Viro, we have the following.

Expectation 0.10. The 3-manifold invariant constructed in a solution to Problem 0.9 is a categorical analogue of the Turaev–Viro quantum invariant.

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